equality in (8) follows from the fact that $\mathbf{j}_{i-1} = 2K - \mathbf{k}_{i-1}$, so $j = 2K - k$, and consequently, $\binom{2K}{j} = \binom{2K}{k}$. Therefore, $\sum_{j=0}^{2K} \binom{2K}{j} (1-p_{i-1})^j p_{i-1}^{2K-j} \left(1 - \frac{1}{N}\right)^{2K-j} = P_{CC,i}$ (see eq. 4).

The second term (the sum over j) in (8) can be expressed in closed form using the identity

$$\sum_{n=0}^{J} \binom{J}{n} n a^n x^{J-n} = Ja(a+x)^{J-1}, \tag{9}$$

which is proven in Appendix A. Applying (9) to (8) results in:

$$P_{CH,i} = P_{CC,i} - \frac{2K\alpha_{i-1}(1-p_{i-1})}{N} \left(1 - \frac{p_{i-1}}{N}\right)^{2K-1} \tag{10}$$

Intuitive interpretation of (10) is less obvious than that of (6), but can be developed as follows. To have a clear hop in SF i, the reference user must randomly select a frequency that satisfies two conditions:

> (a) it is *not* held over by any of the $\mathbf{j}_{i-1}$ overlapping hops that do *not* get hit in SF (i-1) and
>
> (b) it does not get hit by any of the $\mathbf{k}_{i-1}$ overlapping hops that *do* get hit in SF i-1.

Letting $P(a)$ and $P(b)$ denote the respective probabilities of satisfying conditions (a) and (b), then $P(ab) = P(b) - P(\overline{a}b)$. In other words, the set of frequencies that satisfy both conditions (a) and (b) is the set of all frequencies that satisfy (b) minus the set of all frequencies that satisfy (b) but *do not* satisfy (a). Therefore, the probability that a frequency satisfies both conditions (a) and (b) is the probability of satisfying (b) minus the probability of satisfying (b) and *not* satisfying (a). The probability of choosing a frequency that satisfies condition (b), regardless of whether or not condition (a) is satisfied, is simply $P(b) = (1 - p_{i-1}/N)^{2K} = P_{CC,i}$. The desired probability is thus $P_{CC,i}$ minus $P(\overline{a}b)$, which is the probability of selecting a frequency that *is* held over and does not get hit. By Bayes' theorem (see [1], p. 38), $P(\overline{a}b) = P(b\,|\,\overline{a})P(\overline{a})$. The probability of selecting a held-over frequency is $P(\overline{a}) = 2K\alpha_{i-1}(1-p_{i-1})/N$, since $E[\mathbf{j}_{i-1}] = 2K(1-p_{i-1})$ (the expected value of $\mathbf{j}_{i-1}$) and the total number of possible frequencies is N. Given that a held frequency is selected, the probability that it is not hit is $P(b\,|\,\overline{a}) = (1-p_{i-1}/N)^{2K-1}$, because if the reference user selects a frequency held by one overlapping hop, there are $2K$-1 remaining overlapping hops that could hit it. Hence, the probability of selecting a frequency that is not hit but *is* held, is $P(\overline{a}b) = [2K\alpha_{i-1}(1-p_{i-1})/N]\cdot(1-p_{i-1}/N)^{2K-1}$. Subtracting this from $P(b)$ (the *total* probability of selecting a frequency that is not hit) gives (10), which is $P(ab)$, the probability of selecting a frequency that is neither held nor hit.

Using (6) and (10), it is a simple matter to relate p_i to p_{i-1} using

$$\begin{aligned} 1-p_i &= (1-p_{i-1})P_{CC,i} + p_{i-1}P_{CH,i} \\ &= \left(1-\frac{p_{i-1}}{N}\right)^{2K}\left[1-p_{i-1}\frac{2K\alpha_{i-1}(1-p_{i-1})}{N-p_{i-1}}\right]. \end{aligned} \tag{11}$$

C. Approximation of α_i

To complete the model, an expression is needed for α_i. An exact derivation of α_i complicated by the constraints and dependencies that must be taken into account. The statistics of $\mathbf{r}_i$ depend on the value of $\mathbf{j}_i$, and on the way in which the $\mathbf{j}_i$ overlapping "clear" hops are divided between the beginning and end of the reference user's frame. In addition, the overlapping clear hops at the beginning of the reference user's frame must, by definition, all use different frequencies, as must all of the overlapping clear hops at the end of the reference user's frame. Further, clear hops using the same frequency must not overlap in time, and the frame boundaries of the hoppers occur at random times with respect to one another. Finally, the statistics of $\mathbf{r}_i$ will depend not only on $\mathbf{j}_i$, but also on $\mathbf{r}_{i-1}$ and $\mathbf{j}_{i-1}$ (i.e., the system has memory).

Expressions are derived in Appendix B for the mean and variance of $\mathbf{r}_i$ given that $\mathbf{j}_i = j$. This method accounts for all of the effects mentioned above except for the system memory. The results in Appendix B suggest that given $\mathbf{j}_{i-1}$, the standard deviation of $\mathbf{r}_i$ is less than 3% of the mean, and also that $\mathbf{r}_i$ is relatively insensitive to imbalances in the distribution of the $\mathbf{j}_i$ clear hops between the beginning and the end of the references user's frame. Hence, $\mathbf{r}_i$ can be reasonably modeled as a deterministic function of $\mathbf{j}_i$, suggesting that a closed-form expression might be developed for $\mathbf{r}_i$.

To do so, assume that at the beginning of the reference user's frame there are $j/2$ other hoppers with clear frequencies, and at the end of the reference user's frame there also are $j/2$ other hoppers with clear frequencies. Hence, there are $j/2$ "starting frequencies" and $j/2$ "ending frequencies." Some of the ending frequencies *can* be the same as some of the starting frequencies without causing a collision. The objective here is to find what fraction, on average, of the starting frequencies are re-used as ending frequencies.

Let the random variable $\mathbf{t}_h$ represent the fraction of the arbitrary interval that elapses before a unit hops onto a given ending frequency. Assuming $\mathbf{t}_h$ is uniformly distributed between 0 and 1, then if $\mathbf{t}_h$ takes on a particular value t_n for the nth ending frequency, the nth ending frequency can (on average) be the same as $t_n j/2$ of the starting frequencies without a collision. Since the nth ending frequency must be different from the other $j/2 - 1$ ending frequencies, it can be one of $N - j/2 + 1$ frequencies. Therefore, given t_n, the probability that the nth ending frequency is the same as some starting frequency is

$$P\{\text{re-use} \mid \mathbf{t}_h = t_n\} \approx \frac{t_n j/2}{N - j/2 + 1}. \tag{12}$$

Averaging over all possible values of $\mathbf{t}_h$ gives the fraction of ending frequencies that, on the average, are "reused" starting frequencies as:

$$f_r \sim \frac{j/2}{N - j/2 + 1}\int_0^1 t\,dt = \frac{1}{2}\,\frac{j/2}{N - j/2 + 1}. \tag{13}$$

It is clear that $\overline{\mathbf{j}_i}/2 = K(1 - p_i)$. Replacing $j/2$ with $K(1 - p_i)$, α_i is then approximated as:

$$\alpha_i \approx 1 - \frac{f_r}{2} = 1 - \frac{1}{4}\frac{K(1-p_i)}{N-K(1-p_i)+1} . \tag{14}$$

As discussed in Appendix B, this formula gives a result remarkably consistent with the more rigorous analysis. Even closer agreement can be achieved by "tuning" the approximation slightly to be:

$$\alpha_i \approx 1 - \frac{1}{4}\frac{K(1-p_i)}{N-0.9K(1-p_i)} . \tag{15}$$

This formula will be referred to as the "tuned" approximation, and will be used in the calculations that follow. On the graph shown in Fig. 8, this tuned approximation is virtually indistinguishable from the computation of Appendix B.

D. Model Summary

The result of the above development is the recursive relationship

$$p_i = 1 - \left(1 - \frac{p_{i-1}}{N}\right)^{2K}\left[1 - p_{i-1}\frac{2K\alpha_{i-1}(1-p_{i-1})}{N-p_{i-1}}\right], \tag{16}$$

where p_i is the probability that a given user gets hit on a given hop in the ith superframe, K is the number of "interfering" units operating in addition to the reference user, N is the number of available frequencies, and α_{i-1} is approximated as a function of p_{i-1} by (15). Using (16), the probability of a hit can be computed for successive superframes, and as discussed below, some insights into capacity limits can be gained. Clearly, if all units are hop-synchronized, (16) is modified by removing α_i and replacing $2K$ with K.

E. Capacity Implications

Of particular interest is the number of units that can be accommodated by a given number of frequencies with an expectation of eliminating collisions within a relatively short time. This requires that $p_i \ll 1$. If that is the case, (16) can be approximated by:

$$p_i \approx (1 + \alpha_{i-1}) \frac{2Kp_{i-1}}{N} . \tag{17}$$

Clearly, for p_i to continually decrease with increasing i, it is necessary that $K < N/(2 + 2\alpha_i)$. If this condition is met, the hit probability p_i never becomes zero, but becomes arbitrarily small with increasing i (in an actual system, random walk effects would drive the system to a collision-free state). Hence, the "capacity" can be expressed as:

$$K_C \triangleq \frac{N}{2 + 2\alpha} . \tag{18}$$

$$\alpha = 1 - \frac{1}{4} \frac{K_C}{N - 0.9K_C} . \tag{19}$$

Substituting (19) into (18) and solving the resulting quadratic for K_C yields $K_C = N/3.8$ and $\alpha = 0.91$. For $N = 173$ channels (corresponding to channels 150 kHz wide), $K_C = 45$ (so $K_C + 1 = 46$ units total). With synchronized hopping, $K_C = N/2$, so $K_C + 1 = 87$ units for $N = 173$.

If $K > K_C$, then p_i will not asymptotically approach zero but instead will have some steady-state non-zero value. This steady-state hit probability, denoted here by p_{SS}, can be found by substituting p_{SS} for p_i and p_{i-1} in (16) and solving numerically. As shown in table 1, doing so gives non-zero values of p_{SS} for $K > 45$, indicating that the hit probability does not approach zero but rather reaches some lower limit and then remains constant regardless of how many superframes pass. Actually, as will be seen in the next section, K can be slightly greater than K_C, and a no-collision state can still be reached in a moderate number (e.g., several hundred) superframes.

TABLE 1. Steady-State Collision Probability for Frequency-Hopped Cordless Telephones with $N = 173$.

K	p_{SS}
46	0.0210
47	0.0456
48	0.0691
49	0.0917
50	0.1134
51	0.1342
52	0.1542
53	0.1734
54	0.1920
55	0.2099
60	0.2903
65	0.3583
70	0.4166
75	0.4672
80	0.5114
85	0.5504
90	0.5850

III. SIMULATION RESULTS AND RANDOM WALK EFFECTS

A. Summary of Simulation Results

A computer simulation was developed for comparison with the analytic results discussed above. Consistent with the requirements of the FCC Rules, the simulation included the constraint that no frequency can be used more than once in a hopper's sequence, which in the examples discussed here was assumed to consist of 50 frequencies.

Fig. 1 compares the convergence trajectory predicted by the model (eq. 16) to that observed on a sample run of the simulation, for 40 hoppers ($K = 39$). It also was assumed that a frequency replacement is triggered by a collision on a single hop. Obviously, if multiple repeated collisions are required to trigger a replacement, the convergence time would be correspondingly lengthened.

Although it appears from the example in Fig. 1 that agreement between the model and the simulation of the collision trajectory is excellent for $K < K_C$, it was found that the simulation could in fact reach a no-collision state with K larger than K_C (but not very much larger). For example, Fig. 2 shows the average number of collisions per superframe per hopper for 50 hoppers ($K = 49$). As can be seen, a collision-free state is reached in slightly over 300 superframes, although the model predicts a steady-state value of about 4.6 collisions per superframe per hopper, per table 1. It was found from experimentation that the highest number of users for which the simulation reached a no-collision condition within 1000 superframes was 51 ($K = 50$). Fig. 3 shows the simulation results (number of collisions vs. superframe) for this case, for two different simulation sample runs (the random number generator was seeded with two different values). For the first run, a collision-free state was reached after less than 350 superframes, while for the second run, collisions were still occurring at 500 superframes. For both runs, the collision rate was significantly below the level predicted by the model (5.67 collisions per superframe per hopper).

Figs. 4 shows simulation and analysis results for 60 hoppers. Note that for this case, the number of collisions fluctuates reasonably tightly about an average slightly below that predicted by the model.

Fig. 5 shows p_{SS} and the corresponding value of α (labelled α_{SS}) as functions of K for both the model and the simulation. The value of p_{SS} predicted by the model seems to upper-bound that observed in the simulation, and the bound becomes increasingly tight as K increases. The value of α_{SS} predicted by the model seems to have a relatively constant offset from that extracted from the simulation. This probably is because the model for α_i does not account for system memory. However, the effect is very small, and clearly an empirical adjustment for α_i could be incorporated into the model if desired. An additional difference between the mathematical model and the simulation is that while the model considers only successive iterations of a single isolated frame, the simulation inherently includes the effects of dependencies among frames (due to overlap, and the constraint that no hopper can use a given frequency more than once).

Fig. 6 shows simulation and model results for 80 hoppers ($K = 79$) with synchronized hopping (the "capacity" in this case is 87 hoppers). As with the results for unsynchronized hopping, a collision-free state is eventually reached in the synchronized case for a number of hoppers slightly greater than the capacity (e.g., 100 hoppers). For larger numbers of hoppers (e.g., 120), the number of collisions per superframe fluctuates fairly tightly about an average slightly below that predicted by the model.

While the simulation results agree with the model in many respects, the simulation indicates that a collision-free state can in fact be reached with values of K for which p_{SS} is low but non-zero.

These results initially seemed to contradict the analysis of steady-state collision probability presented above, but further development of the analytic model as given below suggests a mechanism for this behavior.

B. The System Viewed as a Markov Chain

The group of frequency-hoppers can be viewed as a system which at any given time is in one of a finite number of discrete states. The state of the system could be defined in various ways. For example, the system state at iteration (superframe) i could be defined as the number of collisions that take place during that iteration. For each state, there is an associated probability that the system is in that state. The state probabilities after i iterations can be expressed as a vector $\mathbf{p}^{(i)}$, the elements of which correspond to the state probabilities; e.g., $\mathbf{p}^{(i)} = [p_0^{(i)}\ p_1^{(i)} \ldots p_M^{(i)}]$, where $p_0^{(i)}$ represents the probability that the system is in state 0 (e.g., no collisions) after i iterations, $p_1^{(i)}$ corresponds to the probability of 1 collision, etc. (although in the particular problem considered here, the probability of exactly 1 collision is always zero). Hence, the elements of $\mathbf{p}^{(i)}$ comprise the state probability distribution after i iterations.

If, after i iterations, the system is in some state m, then there is some probability that after the next iteration, it will be in some other state n. This probability is denoted by p_{mn} and is usually called a state transition probability (in this case, the transition probability from m to n). If the system has M possible states, the set of transition probabilities can be represented as an $M \times M$ matrix $\mathbf{P}$, the elements of which are the transition probabilities; that is, $\mathbf{P}(m,n) = p_{mn}$. If a system can take on a finite number of states, and the state of system after iteration i depends only on its state after iteration i-1, then the system can be modeled as a finite Markov chain. Further, if the matrix of transition probabilities is time-invariant the chain is called *homogeneous*.[3] Both of these conditions hold for the problem of interest here.

Under these conditions, $\mathbf{p}^{(i)}$, the vector of state probabilities after i iterations, is simply:

$$\mathbf{p}^{(i)} = \mathbf{p}^{(0)}\,\mathbf{P}^{i}\,, \tag{20}$$

3. For a concise summary on Markov chains and graphs, see Appendix 6 of [2]. The discussion of Markov chains given here is extracted from that material.

where $\mathbf{p}^{(0)}$ is the initial state distribution, and $\mathbf{P}^i$ is simply the state transition matrix $\mathbf{P}$ raised to the ith power. Thus, if the state transition probabilities and the initial distribution are known, then the distribution after any iteration can be computed.

Viewing the behavior of the group of frequency-hoppers as a Markov chain, the probability that there are no collisions after i iterations is $\mathbf{p}^{(i)}[0]$ (i.e., the zero-th element of the ith iteration of the state probability vector $\mathbf{p}^{(i)}$). It should be noted that in general $\mathbf{p}^{(i)}[0]$ will increase with increasing i, because the system can always go to a no-collision state from other states (although the probabilities may be very small), but cannot go from a no-collision state to any other state, as the problem has been defined here. This observation suggests that, for a given N and K, as i increases, it becomes increasingly likely that a no-collision state will be reached, although if K is much larger than K_C, the growth of $\mathbf{p}^{(i)}[0]$ may be so slow that an extremely large number of iterations would be required to obtain reasonable probabilities of reaching a no-collision state.

C. Analysis of State Transition Statistics

For $K = 50$ and $N = 173$ channels, applying (16) with $p_{SS} = p_i = p_{i\text{-}1}$ gives a steady-state collision probability of about 11% (see table 1), but the simulation indicated that a no-collision state could eventually be reached under these conditions (51 hoppers and 173 channels). This subsection explores that apparent contradiction.

For consistency with the earlier model development, the "system" will be considered a single timeslot (hop number) in the superframe as seen by the reference user, and the state of the system will be defined as the number of overlapping hops that did not experience collisions on iteration i-1 (represented by the random variable $\mathbf{j}_{i\text{-}1}$ in the model). Of interest are the system state statistics after iteration i, given that after iteration i-1 the system is in state j (that is, given that $\mathbf{j}_{i\text{-}1} = j$).

In general, we would like to know the transition probability $P\{\mathbf{j}_i = J \mid \mathbf{j}_{i\text{-}1} = j\}$. One way to find this conditional distribution is to recognize that overlapping hops that are clear after iteration i can be categorized as: (1) those that were clear in SF i-1 and stayed clear, and (2) those that were hit in SF i-1 and became clear in SF i. The distribution for the number that were clear and stayed clear is:

$$P\{\mathbf{n}_{CC} = m \mid \mathbf{j}_{i\text{-}1} = j\} = \binom{j}{m} P_{CC}^m (1 - P_{CC})^{j-m} , \qquad (21)$$

and the distribution for the number that were hit and became clear is:

$$P\{\mathbf{n}_{HC} = n \mid \mathbf{j}_{i-1} = j\} = \binom{k}{n} P_{CH}^{n} (1 - P_{CH})^{k-n} , \tag{22}$$

where $k = 2K - j$ and P_{CC} and P_{CH} are understood to be conditioned on $\mathbf{j}_{i-1} = j$. Thus, $P_{CC} = (1 - 1/N)^k$ and $P_{CH} = P_{CC}(1 - \alpha j/N)$, where α is given as a function of j by the "tuned" approximation of (15), with $j/2$ substituted for $K(1 - p_i)$.

Clearly, $\mathbf{j}_i = \mathbf{n}_{CC} + \mathbf{n}_{HC}$. For purposes of this discussion, $\mathbf{n}_{CC}$ and $\mathbf{n}_{HC}$ will be treated as independent, so the pdf of $\mathbf{j}_i$ (given $\mathbf{j}_{i-1}$) can be modeled as the discrete convolution of the pdfs of $\mathbf{n}_{CC}$ and $\mathbf{n}_{HC}$:

$$P\{\mathbf{j}_i = J \mid \mathbf{j}_{i-1} = j\} = \sum_{m=1}^{j} P\{\mathbf{n}_{CC} = m \mid \mathbf{j}_{i-1} = j\} P\{\mathbf{n}_{HC} = J - m \mid \mathbf{j}_{i-1} = j\}$$

$$= \sum_{m=1}^{j} \binom{j}{m} P_{CC}^{m} (1 - P_{CC})^{j-m} \binom{k}{J-m} P_{CH}^{k-m} (1 - P_{CH})^{k-(J-m)} . \tag{23}$$

Unfortunately, this expression does not appear to be very useful for gaining immediate insights into system dynamics. However, the mean and variance of $\mathbf{j}_i$ given $\mathbf{j}_{i-1}$ are easily found from (21) and (22) as:

$$\eta_j \triangleq E[\mathbf{j}_i \mid \mathbf{j}_{i-1} = j] = E[\mathbf{n}_{CC} \mid \mathbf{j}_{i-1} = j] + E[\mathbf{n}_{HC} \mid \mathbf{j}_{i-1} = j]$$

$$= jP_{CC} + kP_{CH} , \tag{24}$$

and

$$\sigma_j^2 \triangleq VAR[\mathbf{j}_i \mid \mathbf{j}_{i\text{-}1} = j] = VAR[\mathbf{n}_{CC} \mid \mathbf{j}_{i\text{-}1} = j] + VAR[\mathbf{n}_{HC} \mid \mathbf{j}_{i\text{-}1} = j]$$

$$= jP_{CC}(1 - P_{CC}) + kP_{CH}(1 - P_{CH})\,. \tag{25}$$

The terms η_j and σ_j are easily evaluated and can provide useful insights into system behavior. For example, Table 2 shows η_j and σ_j for three cases: $K = 48$, $K = 52$, and $K = 55$. Note that in Table 2, $\eta_j > j$ for $j = 80$ and $j = 85$, and $\eta_j < j$ for $j > 90$. In fact, for a given K there is a "steady-state" value of j for which $n_j = j$. This steady-state value is given by:

$$j_{SS} = 2K(1 - p_{SS})\,. \tag{26}$$

Table 3 shows j_{SS} for K in the range of interest, and Fig. 7 shows j_{SS} and $\alpha_{SS} j_{SS}$ (which represents the number of held-over frequencies) as functions of K for the model and the simulation.

D. Discussion of System Dynamics

From Table 2 a qualitative understanding of system dynamics can be gained. A collision-free state is achieved when j reaches its maximum value of $j_{\max} = 2K$. As shown in Table 3, the statistically "preferred" value of j (j_{SS}) is very near 90 for K not much larger than the "capacity" value K_C found in the previous section (which is about 45 for $N = 173$). To attain a no-collision state, the system must complete a random walk between j_{SS} and $j_{\max} = 2K$.

Note that for $j > j_{SS}$, $\eta_j < j$, and for $j < j_{SS}$, $\eta_j > j$. This means that the transition statistics tend to pull the system state to its expected, or steady-state value, but it is clear from the examples in Table 2 that this bias toward the equilibrium state is not very strong. If, after iteration i-1, the system is in state $j_{i\text{-}1}$, the distribution of $\mathbf{j}_i$ is nearly centered on $j_{i\text{-}1}$, so the state of the system can randomly "walk" among different values with a relatively small bias in favor of the

TABLE 2. Expected value and standard deviation of next state given current state (j), for N = 173 frequencies.

j	$K=48$		$K=52$		$K=55$	
	η_j	σ_j	η_j	σ_j	η_j	σ_j
80	81.24	3.23	81.54	3.88	81.63	4.27
85	85.40	2.77	85.45	3.56	85.37	4.02
90	89.96	2.11	89.75	3.16	89.48	3.71
91	90.92	1.94	90.65	3.06	90.35	3.64
92	91.90	1.75	91.58	2.96	91.23	3.56
93	92.90	1.53	92.52	2.85	92.13	3.48
94	93.92	1.25	93.48	2.74	93.04	3.40
95	94.95	0.89	94.45	2.61	93.98	3.31
96	96.00	0.00	95.44	2.48	94.92	3.22
97			96.45	2.33	95.89	3.12
98			97.47	2.17	96.87	3.02
99			98.52	2.00	97.87	2.91
100			99.58	1.80	98.88	2.79
101			100.66	1.57	99.91	2.66
102			101.76	1.29	100.96	2.53
103			102.87	0.92	102.03	2.38
104			104.00	0.00	103.11	2.22
105					104.22	2.04
106					105.34	1.83
107					106.48	1.60
108					107.63	1.31
109					108.81	0.93
110					110.00	0.00

equilibrium state. Note, however, that as the state (j) approaches its maximum value of $2K$, the standard deviation σ_j shrinks, indicating that state changes become more and more limited to near neighbors. Because of this, if the system "jumps" to a large value of j, it will tend to stay near that value rather than return to the value from which it jumped. This "ratcheting" effect was observed in the simulation for values of K near the "borderline" value of 50; this would

TABLE 3. Equilibrium value of j (j_{SS}).

K	j_{SS}
46	90.21
47	89.85
48	89.50
49	89.14
50	88.79
51	88.43
52	88.09
53	87.74
54	87.38
55	87.03
56	86.68
57	86.32
58	85.97
59	85.62
60	85.27

seem to account for the low values of p_{SS} (relative to those predicted by the model) observed in the simulation for K near 50. Also note that as j approaches $j_{\max} = 2K$, the bias toward the equilibrium state j_{SS} actually *diminishes*, so once the system state reaches this region, it has a reasonable chance of reaching the collision-free state. This effect was also observed in the simulation.

As K becomes larger, it becomes increasingly difficult for the system to "walk" across the middle ground between j_{SS} and values of j near $j_{\max}$. This is because (1) the middle ground expands by two steps each time K increases by 1, (2) the decreases in σ_j with increasing j become slower, which weakens the ratcheting effect, and (3) the bias toward j_{SS} becomes greater. These effects can be seen clearly by comparing the three cases shown in Table 2. In combination, they greatly decrease the likelihood that the system will walk from j_{SS} to $j_{\max}$ in a given number of iterations.

The time required for the system to converge is of course a random variable, but intuitively, its expected value should increase significantly as K increases. Correspondingly, the probability that the system achieves a no-collision state within a given number of iterations will decrease as K increases. Thus, were the simulation to run for a long enough time, we might expect a no-collision state to eventually be reached for $K = 51$, $K = 52$, and so on, but the probability that the system achieves a collision-free state in a reasonable number of iterations would soon become vanishingly small.[4]

IV. THE EFFECT OF WIDEBAND FORWARD LINKS

A. Extending the Model

The model developed here can be extended in a straightforward manner to account for the effect of interference from a wideband forward link (WFL) associated with AVM/LMS (Automatic Vehicle Monitoring/Location and Monitoring Services) systems. Consistent with the development up to this point, it will be assumed that the forward link transmitter is sufficiently close to the cordless telephone to cause interference, and issues of timing, adaptation, and collision probabilities will be analyzed.

Several new parameters need to be introduced to account for the bandwidth and timing of the forward link transmissions. If the forward link has a transmission bandwidth of B_{FL} and the total bandwidth is B_{TOT}, then the fraction of frequency-hopping channels unaffected by the forward link is $\beta \triangleq 1 - B_{FL}/B_{TOT}$. The forward link is assumed to transmit short bursts at random intervals, with some known average burst rate. The frequency hopper is assumed to use some specific bits in its frame to check for a "collision." If any of those bits (or some specified number of them) are corrupted, then the hopper assumes that a collision has occurred and replaces the frequency. It also mutes the frame, to avoid subjecting the user to the corrupted speech codec output. Given the frame structure and the duration of this bit sequence as a fraction of the frame duration, as well as the burst length and the average burst rate for the AVM/LMS forward link, there is some probability ν that the forward link *does not* transmit

4. Without further analysis, definitive statements cannot be made about the relationship between K and the statistics of the convergence time. The problem of interest here is more complex that the classical "coin toss" random walk problem (see p. 290 of [1]), in which the system state (the net distance walked) is the number of "heads" minus the number of "tails." In that case, the system must change state at each iteration (each coin toss), and will "walk" exactly 1 step in one direction or the other. In the problem considered here, the system can transition to one of multiple states, or can remain in the same state. In addition, the transition statistics depend on the state. As a possible area for further work, exploration of the relationship between K and the convergence-time statistics may be of some interest. However, it may prove to be a difficult problem, and the results given here suggest quite clearly the practical limits on K.

during the transmission of the "control" bits in a particular frame, and therefore does not trigger a frequency replacement. Finally, since the WFL transmissions often will not coincide with the control bits, there will be "hits" on the speech bits that are undetected. Consequently, the cordless telephone will not mute the affected frame, nor will a frequency replacement be triggered. These undetected hits can have a significant impact on the link quality perceived by the user. The average number of WFL transmissions per hop during speech bits of frames for which WFL transmissions *do not* coincide with control bits will be denoted by ξ.

For notational convenience, let

$$A_i \triangleq \left(1 - \frac{p_{i-1}}{N}\right)^{2K} \tag{27}$$

$$C_i \triangleq \frac{2K\alpha_{i-1}(1-p_{i-1})}{N-p_{i-1}} \tag{28}$$

In terms of the base case (no wideband forward links), $P_{CC,i} = A_i$, $P_{CH,i} = A_i(1-C_i)$, and $p_i = 1-A_i(1-p_{i-1}C_i)$. As before, (27) and (28) apply for the unsynchronized case; when all hoppers are synchronized, α would be removed and $2K$ would be replaced with K.

There is some probability β_i that a given frequency of a particular hopper *does not* fall within the transmission band of the wideband forward link on the ith SF. For the first SF, this probability is simply $\beta_1 = 1 - B_{FL}/B_{TOT}$, as noted above. As time progresses, it is reasonable to expect that β_i will increase, as the hoppers attempt to migrate away from the forward link. As also noted above, the wideband forward link transmits a burst during the control bits of a particular hopper with probability $1-\nu$. This probability will not change with time. Given ν and β_i, the probability that the wideband forward link *does not* hit the control bits of a particular hopper is

$$G_i \triangleq \beta_i + \nu(1-\beta_i) = \nu + \beta_i(1-\nu)\,. \tag{29}$$

It then follows that

$$P_{CC,i} = A_i G_i \quad (30)$$

and

$$P_{CH,i} = A_i G_i (1 - C_i)\,. \quad (31)$$

As before, $q_i = 1 - p_i = q_{i-1} P_{CC,i} + p_{i-1} P_{CH,i}$, giving

$$q_i = 1 - p_i = A_i G_i (1 - p_{i-1} C_i)\,. \quad (32)$$

Clearly, a recursive expression is also needed for β_i. If a hopper is outside of the WFL band, it can be hit by another hopper, but not by the WFL. If it is hit by a hopper, it will select another frequency that is outside the WFL band with probability β_1. If the hopper is within the WFL band, it can be hit by either a hopper or the WFL, and if it is hit, the probability that the replacement frequency is outside the WFL band is again β_1. Therefore, the probability that a given hop frequency is outside the WFL band in SF i is:

$$\beta_i = \beta_{i-1}(q_{h,i-1} + p_{h,i-1}\beta_1) + (1 - \beta_{i-1})\beta_1 p_{i-1}\,, \quad (33)$$

where $p_{h,i}$ is the probability of being hit by another hopper in SF i and $q_{h,i} = 1 - p_{h,i}$, and is given by:

$$q_{h,i} = q_{i-1} A_i + p_{i-1} A_i (1 - C_i) = A_i (1 - p_{i-1} C_i)\,. \quad (34)$$

The average number of undetected WFL hits per hop is $\xi(1-\beta_i)$. Accounting for the possibility that another hopper can also hit a frame and cause a mute, the net average number of *undetected* hits per hop (i.e., hits on the speech bits without a collision with another hopper or a hit on the control bits from the WFL) is:

$$\overline{n}_{udh,i} = \xi q_{h,i}(1-\beta_i)\,. \tag{35}$$

Note that by the definition of (35), if a frame suffers a detected collision from another hopper or the WFL (a hit on the control bits), any other hits from the WFL during the frame are not counted as "undetected" hits, because the detected collision will cause a frame mute.

Given (27)-(35), p_i, $p_{h,i}$, β_i, and $\overline{n}_{udh,i}$ can be computed recursively for successive superframes. For the first superframe, $q_{h,1} = (1-2/N)^K$, and $q_1 = q_{h,1}G_1$.

B. Results

Before numerical results can be obtained, the parameters ν and ξ must be specified. They will depend on the frame structure of the hopper and the transmission timing of the WFL. It will be assumed that the WFL transmission timing structure is as follows. Each WFL "slot" is 640 μs in duration. During the slot, the base transmits for 200 μs and then waits for a response from the mobile. On average, a given base is assumed to use 3% of the slots (so the overall transmit duty cycle of a given base is 1% on average). Slot usage by a particular base is assumed to occur at random times, so for 3% average usage per base, the probability that any particular slot is *not* used by a given base is 0.97. The frame of the cordless telephone is assumed to be 5 milliseconds in duration. Therefore, the cordless telephone frame spans $5 \div 0.64 = 7.8$ WFL slots, so it overlaps 8 slots. Assuming that the WFL base transmits with the average duty factor, the probability that *none* of the slots overlapping a given cordless telephone frame are used by the WFL base is $P_0 = 0.97^8 = 0.784$. The probability of exactly n hits from the WFL during a frame is $P_n = C_{8,n}\, 0.97^{8-n}\, 0.03^n$. Thus, $P_1 = 0.194$, $P_2 = 0.021$, and $P_3 = 0.0013$. Since $1-\sum_{n=0}^{3} P_n = 5.1\times10^{-5}$, it is clear that the probability of 4 or more hits is sufficiently small to be neglected for purposes of this analysis. The "expected" (average) number of hits per frame is $7.8 \times 0.03 = 0.23$. To calculate ν, it will be assumed that there are two blocks of control bits (one for the handset-to-base link and one for the base-to-handset link). The probability that the WFL does not transmit during either block is $\nu = 0.97^2 = 0.94$. Therefore, there will be WFL transmissions during at least one of the control fields on 6% of the frames, on average, and $\xi = 0.23 - 0.06 = 0.17$.

Fig. 9 shows analytic and simulation results for 40 hoppers, assuming a WFL bandwidth of $B_{FL} = 16\,\text{MHz}$ and a total bandwidth of $B_{TOT} = 26\,\text{MHz}$. Also shown for comparison is the case of Fig. 1, with no WFL interference. The "detected collisions" are those from the WFL as well as other hoppers. The "undetected hits" are from the WFL, and β shows the fraction of the hopping frequencies outside the WFL band. Note that as in the base case, the simulation results are on average slightly "better" than the analytic results (fewer collisions, fewer undetected hits, and slightly higher β). As in the base case, it appears that a "random walk" mechanism is at work, and the disparity is even more pronounced as the number of hoppers decreases. For example, Fig. 10 shows results for 25 hoppers.

C. Multi-Collision Frequency Replacement Triggering

In the cases shown in Figs. 9 and 10, it was assumed that a single hit on the control bits triggered a frequency replacement. While this approach minimizes the time required to react to interference, it tends to make the system react too quickly to transient problems such as might be caused by a momentary deep multipath fade. A better general approach for stability is to introduce some "damping" into the adaptation process by requiring collisions on a given frequency over a number of superframes to trigger a frequency replacement. For a group of hoppers operating at the same hopping rate, or a fixed interference source, the interference will occur on every superframe and therefore will cause the affected frequency to be replaced.

For WFL interference, this is not the case, since the probability that the cordless telephone will receive multiple successive hits from the WFL on the control bits is relatively low. To account for a multiple-collision requirement for a frequency replacement, the model must be further modified. It will be assumed that m sequential collisions are required to trigger a frequency replacement. The superframes will be treated in m-superframe blocks, because hoppers must collide for m superframes before a frequency can be replaced. This approach allows the problem to be solved with minor modifications to the existing model. Specifically, it is the probability of a frequency replacement, rather than simply a collision, that is the key element of the recursion formula, although the collision probability is still of interest. Denoting by r_i the probability of a frequency replacement on a given hop in SF i, two new parameters are defined as:

$$B_i \triangleq \left(1 - \frac{r_{i-m}}{N}\right)^{2K} \tag{36}$$

$$D_i \triangleq \frac{2K\alpha_{i-m}(1 - r_{i-m})}{N - r_{i-m}} \tag{37}$$

The probability that a given hop is clear in SF i given that there was no frequency replacement on that hop in SF $i - m$ is

$$P_{CNR,i} = B_i G_i . \tag{38}$$

The probability that a given hop is clear in SF i given that there *was* a frequency replacement in SF $i - m$ is

$$P_{CR,i} = B_i G_i (1 - D_i) . \tag{39}$$

From (38) and (39) the probability of "no collision" is

$$q_i = 1 - p_i = (1 - r_{i-m}) P_{CNR,i} + r_{i-m} P_{CR,i} = B_i G_i (1 - r_{i-m} D_i) . \tag{40}$$

Note that $q_{h,i} = B_i(1 - r_{i-m} D_i)$ so $q_i = G_i q_{h,i}$ as before.

Let $u_i \triangleq 1 - r_i$ be the probability that there is no frequency replacement in the ith superframe. For a frequency replacement to occur, the control field needs to experience m sequential collisions. The probability that the WFL does not transmit during one of the control fields of a given hopper is ν. For $m = 2$ and $m = 3$, the probability that the WFL does not transmit during the control bits on a particular hop m superframes in a row can be approximated (for $1 - \nu \ll 1$)[5] by:

$$\chi \approx 1 - m(1-\nu)^m . \tag{41}$$

For example, for $\nu = 0.94$ and $m = 2$, $\chi = 0.993$.

Defining

$$\Gamma_i = \beta_i + \chi(1-\beta_i) , \tag{42}$$

the probability that there is *not* a frequency replacement is

$$u_i = 1 - r_i = q_{h,i}\Gamma_i . \tag{43}$$

The expression for β_i becomes:

$$\beta_i = \beta_{i-m}(q_{h,i-m} + p_{h,i-m}\beta_1) + (1-\beta_{i-m})\beta_1 r_{i-m} . \tag{44}$$

Using (38)-(44), p_i, r_i, and β_i can be computed recursively. The initial conditions are $q_{h,1} = (1-2/N)^K$, $q_1 = q_{h,1}G_1$, and $u_1 = q_{h,1}\Gamma_1$.

5. The exact expressions can be derived in a straightforward manner using combinatorial techniques.

For comparison with Fig. 10, Fig. 11 shows simulation and model results for 25 unsynchronized hoppers if two sequential collisions are required to trigger a frequency replacement. Note that agreement between the model (eqs. 36-44) and the simulation is much better than in the single-collision case, suggesting that the two-collision requirement has reduced the random-walk effects.

The numerical results have been limited up to this point to a relatively large number of hoppers (25 or more), unsynchronized hopping, and a forward link bandwidth of 16 MHz. The effects of a reduced number of hoppers, a smaller forward link bandwidth, and hop-synchronization also are of interest. Fig. 12 shows model and simulation results for 6 hoppers with $B_{FL} = 16\,\text{MHz}$, two-hit frequency replacement triggering, and unsynchronized hopping. Fig. 13 shows the same case except $B_{FL} = 8\,\text{MHz}$. Fig. 14 shows the same case as Fig. 13, except the hoppers are synchronized. The "synchronized" case typifies some business environments, in which the cordless base units are colocated in a single housing and frame-synchronized. As is clear from the model development and from comparison of Figs. 1-4 with Fig. 6, without the WFL, performance is better in the synchronized case than the unsynchronized case. However, comparison of Figs. 13 and 14 suggests that in the synchronized case, the hoppers suffer somewhat more from the WFL if they are synchronized. Movement out of the WFL band is somewhat slower, and the variations in both detected collisions and undetected hits are higher. These phenomena are due to the fact that in the synchronized case, a control field hit occurs simultaneously for all hoppers within the WFL band, which is not the case with unsynchronized hoppers. Thus, in the synchronized case, there is a tendency for multiple hoppers to replace frequencies at the same time, generating more collisions among hoppers. Further, undetected hits will be more highly correlated among hoppers in the synchronized case, accounting for the larger variation in the number of undetected hits about its average value.

V. CONCLUSIONS

The analysis presented here provides a model for the performance of multiple frequency-hopping devices operating in close proximity. It was assumed that each unit independently selects its hopping sequence randomly and adapts to collisions by replacing frequencies on which collisions occur with new randomly selected frequencies. Agreement between the mathematical model and the simulation results was found to be remarkably close, considering that the model analyzes only a single isolated frame, and does not account for interaction among hops, nor for the constraint that a hopper can use a given frequency only once in its sequence.

While this analysis did not account for such "real-world" effects as propagation path loss, it provides useful insights into the dynamics of such a situation. The model shows that without hop-synchronization (units do not change frequencies at the same time), the number of units that can simultaneously share a total of N frequencies and converge to a no-collision state using a collision sense/dynamic frequency replacement (CS/DFR) discipline is about $N/3.8$. The simulation results suggest that with values of K *slightly* larger than this, a collision-free state eventually can be reached due to random-walk effects. However, for values of K significantly larger than $N/3.8$, the simulation does not converge to a collision-free state. If the units are hop-synchronized, as will be the case in some business applications, the "capacity" is $N/2$ rather than $N/3.8$.

The model was extended to account for the effect of interference from an AVM/LMS wideband forward link (WFL). It was found that the WFL interference significantly degrades the performance of the hoppers, resulting in undetected, unmuted "hits" to the audio signal and preventing even a small number of hoppers (e.g., 6) from reaching a collision-free state in a reasonable amount of time.

APPENDIX A

This appendix proves the identity:

$$\sum_{n=0}^{J} \binom{J}{n} n a^n x^{J-n} = Ja(a+x)^{J-1}, \tag{A-1}$$

which is used to derive the closed-form expression for $P_{CC,i}$ given in eq. (10).

Letting $\beta \triangleq a/x$ and noting that

$$\binom{J}{n} = \frac{J}{n}\binom{J-1}{n-1}, \tag{A-2}$$

the left-hand side of (A-1) becomes:

$$\sum_{n=0}^{J} \binom{J}{n} n a^n x^{J-n} = x^J \sum_{n=1}^{J} \binom{J}{n} n \beta^n = x^J J \sum_{n=1}^{J} \binom{J-1}{n-1} \beta^n$$

$$= x^J J \beta \sum_{n=1}^{J} \binom{J-1}{n-1} \beta^{n-1} = x^J J \beta \sum_{n=0}^{J-1} \binom{J-1}{n} \beta^n$$

$$= x^J J \beta (1+\beta)^{J-1} = x^J J \frac{a}{x}\left(1+\frac{a}{x}\right)^{J-1} = Ja(a+x)^{J-1}. \tag{A-3}$$

Thus, (A-1) is proven.

APPENDIX B

In this appendix, expressions are derived for the mean and variance of $\mathbf{r}$, which represents the number of distinct frequencies used on overlapping clear hops as a fraction of the total number of overlapping clear hops (see eq. 7).

Assume that two groups of frequencies are randomly chosen from the same alphabet of N frequencies. The first group consists of M distinct frequencies, and the second of L distinct frequencies. Ignoring for the moment the issue of time overlap, it is of interest to determine the probability that the two sets have exactly n frequencies in common. This can be done as follows. The number of combinations of n frequencies belonging to the first set is simply $C_{M,n} \triangleq \frac{M!}{n!(M-n)!}$, which denotes the number of combinations of M items taken n at a time ("M choose n"). Similarly, there are $C_{N-M,L-n}$ combinations of $L-n$ frequencies not belonging to the first set. Since there are $C_{N,L}$ total combinations of L frequencies chosen from the alphabet of N, the probability of choosing a set of L distinct frequencies that has exactly n frequencies in common with the first set of M frequencies is $C_{M,n}C_{N-M,L-n}/C_{N,L}$.

Given that the two sets have exactly n frequencies in common, and hop times are random and uniformly distributed, the probability that none of the n common frequencies overlap in time is 2^{-n}. Hence, the probability that the two sets have n frequencies in common, none of which overlap in time, is

$$P_n = \frac{2^{-n}C_{M,n}C_{N-M,L-n}}{\sum_{n=0}^{MIN\{M,L\}} 2^{-n}C_{M,n}C_{N-M,L-n}}. \tag{B-1}$$

With $j = M + L$ and $r(n) = 1 - n/j$, the first and second moments and the variance of $\mathbf{r}$ (given N, M, and L) are easily determined as

$$\mu_{\mathbf{r}} = E[\mathbf{r}] = \sum_n (1 - n/j)P_n ; \tag{B-2}$$

$$E[\mathbf{r}^2] = \sum_n (1-n/j)^2 P_n \ ; \tag{B-3}$$

$$\sigma_{\mathbf{r}}^2 = E[(\mathbf{r}-\mu_{\mathbf{r}})^2] = E[\mathbf{r}^2] - \mu_{\mathbf{r}}^2 . \tag{B-4}$$

Fig. 8 shows $\mu_{\mathbf{r}}$ and $\sigma_{\mathbf{r}}$ as functions of M, for $M = L = j/2$. The lower dashed line shows the approximate formula for α given in II(C), and the diamonds represent points extracted from the simulation. Although it is cannot be seen from Fig. 8, the table below shows that given j (which in this case is 90), $\mu_{\mathbf{r}}$ and $\sigma_{\mathbf{r}}$ are not very sensitive even to fairly large imbalances between M and L (and large imbalances are very unlikely), and that interchanging M and L leaves $\mu_{\mathbf{r}}$ and $\sigma_{\mathbf{r}}$ unaffected.

M	L	$\mu_{\mathbf{r}}$	$\sigma_{\mathbf{r}}$
45	45	0.915	0.0252
40	50	0.916	0.0251
50	40	0.916	0.0251
35	55	0.919	0.0246
55	35	0.919	0.0246
30	60	0.924	0.0238
60	30	0.924	0.0238
25	65	0.931	0.0226
65	25	0.931	0.0226

Since $\sigma_{\mathbf{r}}$ is very small relative to $\mu_{\mathbf{r}}$ (about 3%), it is clear that given j, $\mathbf{r}$ can be treated as a constant for practical purposes, and hence can be modeled as a function of j as shown in Fig. 8. Further, it is clear from Fig. 8 that the approximate formula in II(C) gives a very close representation of that functional relationship, and can be used to represent the model parameter α_j, and that the "tuned" approximation gives nearly the same results as the "exact" calculation given here, over the relevant range of j.